Math Society Olympiad Question Paper

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Rules and Instructions

- There are 5 questions, under the following titles in this order:
  - Medicinal/Chemical Mathematics
  - Mathematical Economics
  - Mathematical Computer Science
  - Mathematical Physics
  - Pure Mathematics
- The duration of the Olympiad is 5 hours, from 11:00 to 16:00
- You are not allowed access to any sort of calculator for the duration of the exam
- You are not allowed to access any form of the Internet, nor are you allowed to use any form of Artificial Intelligence to help you
- In the second hour, representatives of the Maths Society will come in case you have any questions to ask about the problems

Medicinal/Chemical Mathematics

The molar Gibbs energy of a certain gas is given by

$$G_m = RT \ln p + A + Bp + \frac{1}{2}Cp^2 + \frac{1}{3}Dp^3$$

where A, B, C, and D are constants. Obtain the equation of state of the gas.

Supplementary Information for the problem

Participants may find the following thermodynamic context useful:

- Gibbs Energy (G): A thermodynamic potential. For a pure substance, its value can be expressed as a function of temperature and pressure.
The **molar Gibbs energy (G_m)** is the Gibbs Energy per mole of substance.

- Equation of State: A constitutive equation that describes the relationship between the state variables of a system. For a gas, this is typically a mathematical relationship linking its **Pressure (p)**, **Molar Volume (V_m)**, and **Temperature (T)**. The objective is to derive such a relationship from the given expression for G_m.

- Fundamental Property Relation: Changes in the molar Gibbs energy for a pure substance are governed by the following fundamental differential relation:

$$dG_m = -S_m dT + V_m dp$$

where the change in molar entropy is equal to the negative change of molar entropy with temperature plus change in molar volume with pressure

Mathematical Economics

Consider two consumers and two commodities. Let q_{ij} denote the quantity of good j allocated to consumer i. The utility (or "happiness") functions for the consumers are:

$$U_1 = q_{11}q_{12} \quad \text{and} \quad U_2 = q_{21}q_{22}.$$

The total available quantities of the goods are q_1 and q_2, so the resource constraints are:

$$q_{11} + q_{21} = q_1 \quad \text{and} \quad q_{12} + q_{22} = q_2.$$

For any pair of fixed utility targets (U_1^0, U_2^0), the **Scitovsky contour** is defined as the set of all aggregate bundles (q_1, q_2) that are just sufficient to achieve these utility levels through some allocation $(q_{11}, q_{12}, q_{21}, q_{22}) \geq 0$ satisfying the constraints. On this contour, the allocation is efficient, which implies the condition:

$$\frac{q_{12}}{q_{11}} = \frac{q_{22}}{q_{21}}.$$

Show that the Scitovsky contour is given by the equation:

$$q_1 q_2 = \left(\sqrt{U_1^0} + \sqrt{U_2^0}\right)^2.$$

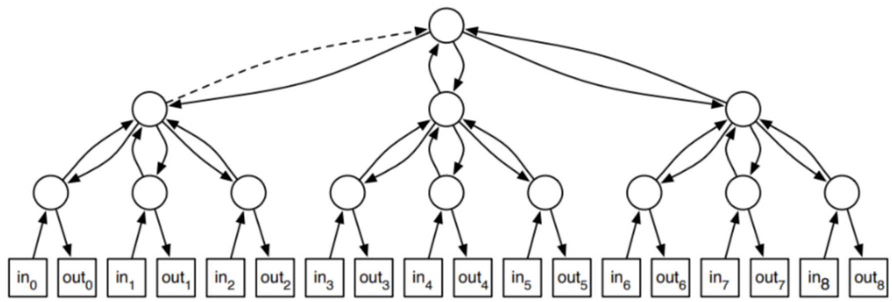

Figure 1: Reference for Mathematical Computer Science

Mathematical Computer Science

Consider the complete ternary-tree network with 9 inputs and 9 outputs shown below, where packets are routed randomly. The route each packet takes is the shortest path between input and output. Let I_0, I_1, and I_2 be indicator random variables for the events that a packet originating at in_0, in_1, and in_2, respectively, crosses the dashed edge in the figure. Let $T = I_0 + I_1 + I_2$ be a random variable for the number of packets passing through the dashed edge.

(a) Suppose that each input sends a single packet to an output selected uniformly at random; the packet destinations are mutually independent. (Note that outputs may receive packets from multiple inputs, including their corresponding input.)

What are the expectation and variance of T?

(b) Now consider the situation where a permutation of inputs to outputs is chosen uniformly at random; each input sends a packet to a distinct output. What is the expected value of T? Briefly justify your answer.

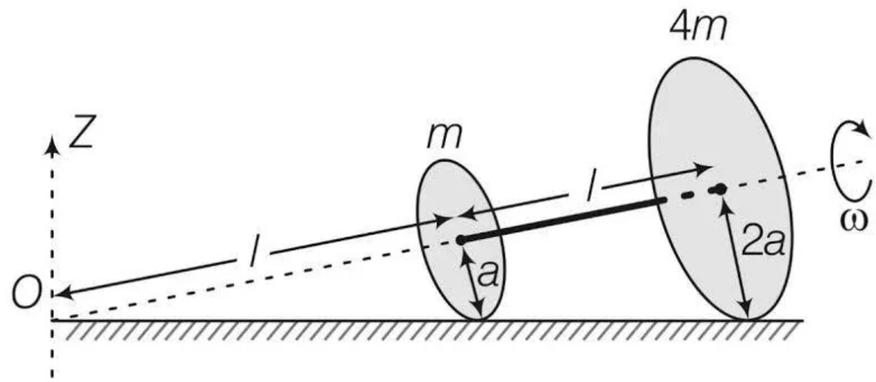

Figure 2: Reference for Mathematical Physics

Mathematical Physics

Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, rigid rod of length $l = \sqrt{24}\,a$ through their centers. This assembly is laid on a firm and flat surface, and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\vec{L}$

(a) Show that the center of mass of the assembly rotates about the z-axis with an angular speed of $\omega/5$

(b) Show that The magnitude of angular momentum of the assembly about its center of mass is $\frac{17}{2}ma^2\omega$

Supplementary Information Regarding Mathematical Physics

1. Moment of Inertia (I)

- **Definition**: Resistance to rotational acceleration
- **General Formula**:

$$I = \int r^2 \, dm \quad \text{or} \quad I = \sum_i m_i r_i^2$$

- **Common Objects**:

Point mass:	$I = mr^2$
Solid disk/cylinder:	$I = \frac{1}{2}MR^2$
Hoop:	$I = MR^2$
Solid sphere:	$I = \frac{2}{5}MR^2$
Rod about center:	$I = \frac{1}{12}ML^2$
Rod about end:	$I = \frac{1}{3}ML^2$

2. Angular Speed (ω)

- **Definition**: Rate of angular displacement
- **Formulas**:
$$\omega = \frac{d\theta}{dt}, \quad \omega = \frac{2\pi}{T}, \quad \omega = 2\pi f$$
- **Relation to linear speed**:
$$v = \omega r$$
- **Units**: rad/s

3. Angular Momentum ($\vec{L}$)

- **Definition**: Rotational analog of linear momentum
- **Point particle**:
$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$
- **Rigid body about fixed axis**:
$$L = I\omega$$
- **General rigid body**:
$$\vec{L} = I\vec{\omega}$$
- **Conservation**: If $\tau_{\text{ext}} = 0$, then $\vec{L}$ is constant
- **Torque relation**:
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Key Relationships

Rotational KE:	$K = \frac{1}{2} I\omega^2$
Torque:	$\tau = I\alpha = rF\sin\theta$
Newton's 2nd Law (rotation):	$\tau = \frac{dL}{dt}$
Parallel Axis Theorem:	$I = I_{\text{cm}} + Md^2$

Pure Mathematics

Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x,y)P(z,t) = P(xz - yt, xt + yz).$$